ISSN:0976-0172

Journal of Bioscience And Technology
www.jbstonline.com

# Nonlinear Self-Adjointness Approach to Invariant Solutions of the 2D Rossby Wave Equation

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***Article Info***

*Received: 27-04-2022* *Revised: 15-07-2022* *Accepted: 26-07-2022*

**Abstract:**

The paper investigates the nonlinear self-adjointness of the nonlinear inviscid barotropic nondivergent vorticity equation in a beta-plane. It is a particular form of Rossby equation which does not possess variational structure and it is studied using a recently method developed by Ibragimov. The conservation laws associated with the infinite-dimensional symmetry Lie algebra models are constructed and analyzed. Based on this Lie algebra, some classes of similarity invariant solutions with nonconstant linear and nonlinear shears are obtained. It is also shown how one of the conservation laws generates a particular wave solution of this equation.

**Introduction**

The concepts of symmetry, invariants and conservation laws are fundamental in the study of dynamical systems, providing a clear connection between the equations of motion and their solutions. There are many reasons for computing symmetries and conservation laws corresponding to systems described by differential equations. In recent years, a remarkable number of mathematical models occurring in various research domains have been studied from the point of view of symmetry group theory [1–3]. The Lie symmetry approach is now an established route for the reduction of differential equations. The method centers on the algebra of one parameter Lie group of transformations admitted by the PDEs. Once known, the reduction of the PDE is standard and may lead to exact (symmetry invari- ant) solutions [4–8]. There are a number of reasons to find conserved densities of PDEs. Some conservation laws are physical (e.g., conservation of momentum, mass, energy, electric charge) and others facilitate analysis of the PDE and predict integrability. Conservation laws play an important role in the development of soliton theory, in the theory of non-classical transformations [9], [10] and in the theory of normal forms and asymptotic inerrability [11]. The knowledge of conservation laws is also useful in the numerical integration of PDEs [12, 13], for example, to control numerical errors. Although No ether's approach provides an elegant algorithm for finding conservation laws, it possesses a strong limitation: it can only be applied to equations that have variational structure. Finding methods for constructing conservation laws for equations without variational structure has been subject of intense research. For example, in [14] a nice relationship is established between symmetries and conservation laws for self-adjoint differential equations, an identity which does not depend on the use of a Lagrangian. Another interesting result concerns a direct link between the components of a conserved vector for an arbitrary partial differential equation and the LieBäcklund symmetry generator associated to the conserved vector's components [15]. Recently, [16] demonstrated a new algorithm for finding conserved vectors associated to any symmetry of nonlinear self-adjoint evolutionary equation. Extensive research has been carried out in

ISSN:0976-0172

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order to find self-adjoint and quasiself-adjoint classes of equations and their conservation laws. For example, the necessary and sufficient conditions for a general fourth-order evolution equation to be selfadjoint is determined in [17], the quasi-self-adjointness of a generalized Camassa-Holm equation was obtained in [18], a quasi self-adjointness classification of quasilinear dispersive equations was carried out in [19]. The purpose of this paper is to apply the recent Ibragimov's approach to the study of two-dimensional Rossby waves. The study of Rossby waves is one of the basic important problems in geophysical fluid dynamics such as atmospheric and oceanic circulation dynamics [20–22]. The Rossby waves studied in this paper are restricted on the following dimensionless inviscid barotropic nondivergent vorticity equation in a beta-plane [23]:

$$\Delta u_t + J(u, \Delta u) + \beta u_x = 0, \qquad (1)$$

where $u$ is the dimensionless stream function, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the 2−dim Laplacian operator, $J(u, \Delta u) = u_x \Delta u_y - u_y \Delta u_x$ is the Jacobian. We shall characterize the Earth's rotation effects through the quantity $\beta = \frac{\omega_0}{R_0}\cos(\phi_0)(\frac{L^2}{U})$, where $R_0$ is the Earth's radius, $\omega_0$ is the angular frequency of the Earth's rotation, $\phi_0$ is the latitude, $L$ and $U$ are the characteristic horizontal length and velocity scales.

Neglecting the effects of the Earth's rotation (β = 0), the general fluid dynamics can be described by the Navies Stokes equation. In [24] some exact solutions of the Navies-Stokes equation have been found from the symmetry group analysis. Because of the non-inerrability and of high nonlinearity of Eq. (1), one usually studies the Ross by waves numerically or approximately [25]. This paper is organized as follows. The essential points of Ibragimov's method will be summarized and the nonlinear self-adjointness of evolution equation (1), the most important point of applying Ibragimov's method, will be investigated in Section 2. The conservation laws provided by infinite-dimensional symmetry Lie algebra admitted by the 2D Rossby equation will be constructed in Section 3. The next section of the paper will illustrate the algorithm for constructing invariant solutions of our model with respect to some one-dimensional subalgebras of the whole Lie algebra. New such invariant solutions and a periodic solution provided by the non-trivial conservation law will be point out, respectively. Some concluding remarks will end the paper.

## 2. Nonlinear self-adjointness

There are many interesting results concerning the correspondence between symmetries and conservation laws. Because a large number of differential equations without variational structure admits conservation laws, an intense research has been devoted to find methods for constructing conservation laws for equations without variational structure. In this section we shall present Ibragimov's method [16] which provides an elegant algorithm for finding conserved vectors which can be applied for any differential equation (or systems of equations)

### 2.1. Ibragimov's method

Let us consider a partial differential equation

$$F = F(x, u, u_{(1)}, \ldots u_{(n)}) = 0, \qquad (2)$$

where F is a differential function, x = (x 1 , ..., xn ) are the independent variables, the dependent variable is u = u(x) and u(n) is the set of all partial derivatives of u, up to n-th order.

The formal Lagrangian is introduced by the relation:

$$\mathcal{L} = vF. \qquad (3)$$

It involves a new dependent variable v, the so-called nonlocal variable. It is a similar approach as the use of ghost type variables [27]. Then, the adjoint equation of (2) is defined by

$$F^*(x, u, v, .., u_{(n)}, v_{(n)}) \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \qquad (4)$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}}, \qquad (5)$$

is the Euler-Lagrange operator,

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + u_{ij} \frac{\partial}{\partial u_j} + v_{ij} \frac{\partial}{\partial v_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + v_{ijk} \frac{\partial}{\partial v_{jk}} + \ldots \qquad (6)$$

is the total derivative operator with respect to x i , i, j, k = 1, . . . , n, and summation over repeated indices is assumed.

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The equation F = 0 is said to be nonlinearly self-adjoint if there exists a function

$$v = \varphi(x, u) \tag{7}$$

such that

$$F^* |_{v=\varphi(x,u)} = \lambda F \tag{8}$$

for some undetermined coefficient λ. If v = φ(u) in (7) and (8), Eq. (2) is called quasi-selfadjoint. If v = u, the Eq. (2) is called strictly self-adjoint. Supposing that Eq. (2) is nonlinearly self-adjoint, then applying Ibragimov's theorem to system (2), (4) with the formal Lagrangian (3), one obtains that any Lie point, contact, generalized or nonlocal symmetry

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} \tag{9}$$

admitted by (2) determines a conservation law DiC i = 0 for (2) with the components of the conserved vector given by

$$\begin{aligned} C^i = & \xi_i \mathcal{L} + W \left[ \frac{\partial \mathcal{L}}{\partial u_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) - \ldots \right] \\ & + D_j(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) + \ldots \right] \\ & + D_j D_k(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}} - \ldots \right] + \ldots \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} = & -D_x \left( \frac{\partial \mathcal{L}}{\partial u_x} \right) - D_y \left( \frac{\partial \mathcal{L}}{\partial u_y} \right) - (D_x^2 + D_y^2) \left[ D_t \left( \frac{\partial \mathcal{L}}{\partial \Delta u_t} \right) + D_y \left( \frac{\partial \mathcal{L}}{\partial \Delta u_y} \right) + D_x \left( \frac{\partial \mathcal{L}}{\partial \Delta u_x} \right) \right] = \\ & - D_x[v(\Delta u_y + \beta)] + D_y[v \Delta u_x] - (D_x^2 + D_y^2)[D_t(v) + D_y(v u_x) - D_x(v u_y)]. \end{aligned}$$

After appropriate calculations, the adjoint equation is written as

$- vt_{(2x)} - vt_{(2y)} - ux\ v_{(2x)y + v(3y)} + uy_{[v(2y)x + v(3x)\ ]} - \beta v_x + 2u_{xy}v2x - 2v_{xy}u2x = 0.$ (12)

It is easy to verify that this equation becomes the 2D Rossby wave equation (1) multiplied with constant coefficient λ = −1, upon the substitution v = u.It means that the equation (1) is nonlinearly self-adjoint, specifically it is strictly self-adjoint

Conservation laws provided by Lie point symmetrie

Noether's theorem cannot be directly applied to obtain conservation laws on the basis of the equation's symmetries. This can be overcome by applying the general concept of nonlinear self-adjointness developed by Ibragimov which enables to establish the conservation laws for any differential equation.

Lie point symmetries of the 2D Rossby wave equation.

The Lie algebra of the infinitesimal symmetries of the twodimensional Rossby wave equation (1) has been obtained in [26]. It involves two arbitrary functions of t and contains the following basis of symmetry operators:

$$\begin{aligned} X_1 &= t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 3u\frac{\partial}{\partial u}, \ X_2 = \frac{\partial}{\partial t}, \ X_3 = \frac{\partial}{\partial y} \\ X_f &= f(t)\frac{\partial}{\partial x} - \left( \frac{df(t)}{dt} y \right) \frac{\partial}{\partial u}, \ X_g = g(t)\frac{\partial}{\partial u}. \end{aligned} \tag{13}$$

When the Lie algebra is computed, the following nonvanishing relations are obtained:

$$\begin{aligned} & [X_1, X_2] = -X_2, \ [X_1, X_3] = X_3, \ [X_1, X_f] = X_f + X_{(t\ \dot{f})}, \\ & [X_1, X_g] = X_{(t\dot{g}+3g)}, \ [X_2, X_f] = X_{\dot{f}}, [X_2, X_g] = X_{\dot{g}}, \\ & [X_3, X_f] = X_{(-\dot{f})}. \end{aligned} \tag{14}$$

3.2. Conservation laws associated with symmetries We will apply formula (10) for constructing the conserved vector associated with the symmetries (13) admitted by the 2D Rossby wave equation. Since the maximum order of derivatives involved in formal Lagrangian (11) is equal to three, this formula becomes:

$$\begin{aligned} C^i = & W \left[ \frac{\partial \mathcal{L}}{\partial u_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) \right] \\ & + D_j(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) \right] + D_j(W) D_k(W) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}} \right]. \end{aligned} \tag{15}$$

where the Lagrangian containing mixed derivatives should be written in the symmetric form

ISSN:0976-0172
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$$\mathcal{L} = \frac{v}{3}\{3\beta u_x + u_{t(2x)} + u_{xtx} + u_{xxt} + u_{t(2y)} + u_{yty} + u_{yyt}$$
$$+u_x[u_{(2x)y} + u_{xyx} + u_{yxx} + 3u_{3y}] \qquad (16)$$
$$-u_y[u_{(2y)x} + u_{yxy} + u_{xyy} + 3u_{3x}]\}.$$

Invoking that the analyzed Eq. (2) is strictly self-adjoint with the substitution v = u, we will replace in C i the nonlocal variables v with u, thus arriving to local conserved vectors for Rossby wave equation. Let us apply the procedure to C 1 . Consequently, the density of the conservation law is written in the following

form:

$$C^1 = W\left[D_{2x}\left(\frac{\partial\mathcal{L}}{\partial u_{t(2x)}}\right) + D_{2y}\left(\frac{\partial\mathcal{L}}{\partial u_{t(2y)}}\right)\right]$$
$$-D_x(W)D_x\left(\frac{\partial\mathcal{L}}{\partial u_{t(2x)}}\right) - D_y(W)D_y\left(\frac{\partial\mathcal{L}}{\partial u_{t(2y)}}\right)$$
$$+D_{2x}(W)\left(\frac{\partial\mathcal{L}}{\partial u_{t(2x)}}\right) + D_{2y}(W)\left(\frac{\partial\mathcal{L}}{\partial u_{t(2y)}}\right)$$

or

$$C^1 = \frac{1}{3}\{W(u_{2x} + u_{2y}) - u_x D_x(W) - u_y D_y(W)$$
$$+ u\left[D_{2x}(W) + D_{2y}(W)\right]\}. \qquad (17)$$

Dilation group

Consider the generator of the dilation group from the basis of operators (13), namely:

$$X_1 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 3u\frac{\partial}{\partial u}. \qquad (18)$$

It is interesting to note that the symmetry operator X1 leaves invariant the action attached to the formal Lagrangian (3). This assertion can be easily checked using the Lie equations associated to X1. For this operator, the Lie characteristic has the form:

$W = -3u - tu_t + xu_x + yu_y$. (19)

The substitution of (19) in (17) yields:

$C^1 = \frac{1}{3}\{-tu[u_{t(2x)} + u_{t(2y)}] + t[u_x u_{tx} + u_y u_{ty}] + u_{[xu(3x)} + y_{u(3y)}) + xu_{x(2y)} + yu_{(2x)y]} - tut\,[u_{2x} + u_{2y}] + yu_{2x}u_y + xu_{2y}u_x - (yu_x + xu_y)ux_y - 4u(u_{2x} + u_{2y}) + 2(u^{2x} + u^{2y})\}$. (20)

We modify (20) by using the identities:

$$-tu_t u_{(2x)} = D_x[-tu_t u_x] + tu_x u_{tx},$$
$$-tu_t u_{(2y)} = D_y\left[-tu_t u_y\right] + tu_y u_{ty}.$$

$$tu_x u_{tx} = D_x[tuu_{tx}] - tuu_{t(2x)}, \quad tu_y u_{ty} = D_y[tuu_{ty}] - tuu_{t(2y)}.$$

$$uxu_{3x} = D_x[uxu_{2x}] - \frac{1}{2}D_x(xu_x^2) - D_x(u_x u) + \frac{3}{2}u_x^2.$$
$$uyu_{3y} = D_y[uyu_{2y}] - \frac{1}{2}D_y(yu_y^2) - D_y(u_y u) + \frac{3}{2}u_y^2.$$

Based on the commutativity of the total differentiations, the conserved vector $C = (C^1, C^2, C^3)$ can be reduced to the form:

$$\tilde{C} = (\tilde{C}^1, \tilde{C}^2, \tilde{C}^3)$$

with the components

$$\tilde{C}^1, \; \tilde{C}^2 = C^2 + D_t(h^2), \; \tilde{C}^3 = C^3 + D_t(h^3). \qquad (24)$$

***Remark 2.***
The two-dimensional vector $(\tilde{C}^2, \tilde{C}^3)$ defines the flux of the conservation laws. In the following, we will ignore the tilde in the final expressions of quantities (24).

The conservation law $D_i C^i = 0$ is trivial if and only if its density $C^1$ evaluated on the solutions of Eq. (1), i.e. the quantity $C^1_* = C^1\,|_{(1)}$satisfies the variational derivative:

$$\frac{\delta C^1_*}{\delta u} = 0. \qquad (25)$$

Using the previous statement, let us verify if the analyzed conservation law is a nontrivial one.

$$\frac{\delta C^1_*}{\delta u} = \frac{\delta}{\delta u}[tuJ(u,\Delta u) + \beta tuu_x + 3(u_x^2 + u_y^2)] = -6\Delta u \neq 0. \qquad (26)$$

ISSN:0976-0172
Journal of Bioscience And Technology
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The flux of the conserved vector could be obtained either by means of relations (15) or by proving that the $D_t(C^1)$ evaluated on the solutions of Eq. (1) satisfied:

$$D_t(C^1)\,|_{(1)} = D_x(P^2) + D_y(P^3) \qquad (27)$$

with certain functions $P^2$, $P^3$.

We choose to derive the flux $(C^2, C^3)$ of conserved vector with known density (22). In fact the relation (27) could be really obtained using the master Eq. (1) and the identities:

$$D_t(C^1)\,|_{(1)} = \{(-u - tu_t)(\Delta u_t) - tuD_t(\Delta u_t) + 6[u_x u_{xt} + u_y u_{yt}]\}\,|_{(1)}, \qquad (28)$$

$$uu_x D_y(\Delta u) = D_x\left[\frac{1}{2}u^2 D_y(\Delta u)\right] - \frac{1}{2}u^2 D_{xy}(\Delta u),$$

$$-uu_y D_x(\Delta u) = -D_y\left[\frac{1}{2}u^2 D_x(\Delta u)\right] + \frac{1}{2}u^2 D_{xy}(\Delta u),$$

$$tu_t u_x D_y(\Delta u) = D_x\left[tuu_t D_y(\Delta u)\right] - tuu_{tx} D_y(\Delta u) - tuu_t D_{xy}(\Delta u),$$

$$-tu_t u_x D_x(\Delta u) = -D_y\left[tuu_t D_x(\Delta u)\right] + tuu_{ty} D_x(\Delta u) + tuu_t D_{xy}(\Delta u),$$

$$D_x[t\beta uu_t] = t\beta u_x u_t + t\beta uu_{tx}, \;\; \beta u_x u = D_x\left[\frac{1}{2}\beta u^2\right],$$

$$D_t(\Delta u_t)\,|_{(1)} = -u_{xt} D_y(\Delta u) - u_x D_{ty}(\Delta u) + u_{ty} D_x(\Delta u) + u_y D_{tx}(\Delta u) - \beta u_{xt},$$

$$tuu_x D_{ty}(\Delta u) = D_x\left[\frac{1}{2}tu^2 D_{ty}(\Delta u)\right] - \frac{1}{2}tu^2 D_{xyt}(\Delta u),$$

$$-tuu_y D_{tx}(\Delta u) = D_y\left[-\frac{1}{2}tu^2 D_{tx}(\Delta u)\right] + \frac{1}{2}tu^2 D_{xyt}(\Delta u),$$

$$[u_x u_{xt} + u_y u_{yt}]\,|_{(1)} = D_x\left[uu_{xt} + \frac{1}{2}u^2 D_y(\Delta u) + \frac{1}{2}\beta u^2\right] + D_y\left[uu_{yt} - \frac{1}{2}u^2 D_x(\Delta u)\right]. \qquad (29)$$

Substituting the expressions (29) in the extended right-hand side of Eq. (28), we arrive at Eq. (27) with

$$P^2 = \left[\frac{7}{2}u^2 + tuu_t\right][\Delta u_y + \beta] + \frac{1}{2}tu^2(\Delta u_{ty}) + 6uu_{xt}, \quad (30)$$

$$P^3 = -\left[\frac{7}{2}u^2 + tuu_t\right](\Delta u_x) - \frac{1}{2}tu^2(\Delta u_{tx}) + 6uu_{yt}. \quad (31)$$

In conclusion, denoting $C^2 = -P^2$, $C^3 = -P^3$, the infinitesimal symmetry $X_1 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 3u\frac{\partial}{\partial u}$ from the basis (13) admitted by Eq. (1), provides the conserved vector $C = (C^1, C^2, C^3)$ with the components:

$$C^1 = -tu(\Delta u_t) + 3(\nabla u)^2,$$

$$C^2 = -\left[\frac{7}{2}u^2 + tuu_t\right][\Delta u_y + \beta] - \frac{1}{2}tu^2(\Delta u_{ty}) - 6uu_{xt},$$

$$C^3 = \left[\frac{7}{2}u^2 + tuu_t\right](\Delta u_x) + \frac{1}{2}tu^2(\Delta u_{tx}) - 6uu_{yt}. \quad (32)$$

The quantities (32) do not have a direct physical significance, but they can generate interesting solitary wave solutions of (1), as it will be exemplified in subsection 4.2.

3.2.2. Translation group The one-parameter group of translations in the variables t and x is generated by the operators $X_2$, $X_3$ from the basis (13). We analyzed the conservation laws generated by the invariance of Eq. (1) under this group of symmetries.

(i) Time translation For the operator $X\frac{\partial}{t} =$ , the Lie characteristic is:

W = −ut

Substituting it in (17) and after some appropriate calculations, the final expression for density of the local conserved vector takes the form:

ISSN:0976-0172

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$$C^1 = -u(\Delta u_t) + D_x\left[\frac{2}{3}uu_{tx} - \frac{1}{3}u_xu_t\right] + D_y\left[\frac{2}{3}uu_{ty} - \frac{1}{3}u_yu_t\right]. \quad (34)$$

Dropping the divergent type terms, it gives:

$$C^1 = -u(\Delta u_t). \quad (35)$$

According to Lemma 1, we have to evaluate (35) on the solutions of Eq. (1). Then, we have

$$\frac{\delta C_*^1}{\delta u} = \frac{\delta}{\delta u}[uJ(u,\Delta u) + \beta uu_x] = 0. \quad (36)$$

Hence, the invariance of Eq. (1) under the time translation only provides a trivial conservation law. (ii) Translation of y For the operator $X_3 = \frac{\partial}{\partial y}$, the Lie characteristic is

W = −uy. (37)

Replacing this expression in (17), one can rewrite the flux in the equivalent form:

$$C^1 = D_y\left[\frac{1}{3}\left(\frac{(\nabla u)^2}{2} - u(\Delta u)\right)\right]. \quad (38)$$

Definition 4. The conservation law is said to be trivial if its density C 1 evaluated on the solutions of Eq. (1) is the divergence:

Hence, the invariance of Eq. (1) under the translation of y only provides a trivial conservation law. 3.2.3. Infinite symmetry Lie group Similar calculations show that the symmetry operators $X_f$ and $X_g$ from (13), which involves two arbitrary functions of time, also give trivial conservation laws, in according with the previous condition (39). More exactly, in these two cases, the densities of conservation laws admit, respectively the divergence expressions:

$$X_f \to C^1 = D_x\left[\frac{1}{3}\left(f(t)\left(\frac{(\nabla u)^2}{2} - u(\Delta u)\right) - \dot{f}(t)yu_x\right)\right] + D_y\left[\frac{1}{3}\dot{f}(t)(2u - yu_y)\right]$$

$$X_g \to C^1 = D_x\left[\frac{1}{3}g(t)u_x\right] + D_y\left[\frac{1}{3}g(t)u_y\right]. \quad (40)$$

4. Types of solutions

Let us present some symmetry reductions and associated invariant solutions for underlying Eq. (1). 4.1. Invariant solutions based on symmetry transformations

For a start, let us derive the invariant solution generated by the invariance of the analyzed equation to the dilation group. We will use the assertion that the function u = Ψ(1)(t, x, y) is a group invariant solution of (1) if:

$$X_1[u - \Psi^{(1)}(t,x,y)]\,|_{u=\Psi^{(1)}} = 0, \quad (41)$$

where the operator X1 is provided by (13) This condition is equivalent to the partial differential equation:

$$3\Psi^{(1)} + t\,\Psi_t^{(1)} - x\Psi_x^{(1)} - y\Psi_y^{(1)} = 0, \quad (42)$$

which admits the solution:

$$\Psi^{(1)}(t,x,y) = \frac{H(xt,yt)}{t^3}. \quad (43)$$

Introducing the invariant similarity variables $w = xt$ and $z = yt$ and substituting (43) in Eq. (1), one arrives at the following reduced equation for $H(w,z)$:

$$w\Delta H_w + z\Delta H_z - \Delta H + J(H,\Delta H) + \beta H_w = 0, \quad (44)$$

with $\Delta = \partial^2/\partial w^2 + \partial^2/\partial z^2$. It has the solution:

$$H(w,z) = -\frac{\beta z^3}{6} + \frac{c_3(z^2 - w^2)}{2} + c_4z + c_1w + c_2, \quad (45)$$

where $c_i$, $i = \overline{1,4}$ are arbitrary constants.
Coming back to the original variables, the invariant solution of (1) is given, in this case, by the expression:

$$u(t,x,y) = -\frac{\beta y^3}{6} + \frac{c_3(y^2 - x^2)}{2t} + \frac{1}{t^2}(c_4y + c_1x) + \frac{c_2}{t^3}. \quad (46)$$

For a fixed moment of time $t = 10$ and for the choices of the other constants $\beta = 10^{-11}$, $c_1 = 0.5$, $c_2 = 1$, $c_3 = 2$, $c_4 = 1$, this solution has the form represented in Figure 1.

$$u(t,x,y) = -\frac{\beta y^3}{6} + \frac{c_3(y^2 - x^2)}{2t} + \frac{1}{t^2}(c_4 y + c_1 x) + \frac{c_2}{t^3}. \quad (46)$$

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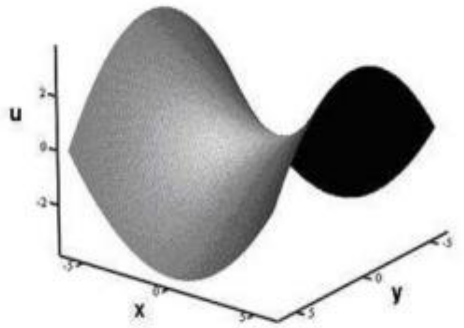


Figure 1. A plot of the Rossby wave described by (46) with $\beta = 10^{-11}$, $c_1 = 0.5$, $c_2 = 1$, $c_3 = 2$, $c_4 = 1$ at time $t = 10$.

We will construct the invariant solution generated by the symmetry operator $(X_3 + X_f) = \frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial x} - \left(\frac{df(t)}{dt} y\right)\frac{\partial}{\partial u}$.

Following the same procedure, the invariance condition is equivalent with the following partial differential equation:

$$\Psi_y^{(2)} + f(t)\Psi_x^{(2)} + \dot{f}(t) y = 0. \quad (47)$$

The solution takes the form:

$$\Psi^{(2)}(t,x,y) = -\dot{f}\,\frac{x}{f}\left(y - \frac{x}{2f(t)}\right) + G\left(t, y - \frac{x}{f(t)}\right). \quad (48)$$

Replacing the previous solution in Eq. (1), one obtains the reduced equation:

$$f^2\ddot{f} - 2f\dot{f}^2 - f\dot{f}(1 + f^2) z G_{(3z)} + f^2(1 + f^2) G_{v(2z)} - 2f\dot{f}\, G_{(2z)} - \beta f^3(\dot{f} z + G_z) = 0, \quad (49)$$

where the invariants are denoted by $w = t$ and $z = y - \frac{x}{f(t)}$.
If we choose a concrete form for the arbitrary function, namely $f(w) = -2c_1 w + c_2$, $c_1, c_2 = const.$, the Eq. (49) generates the solution:

$$G(w,z) = c_1 z^2 + \rho(w), \quad (50)$$

with arbitrary function $\rho(w)$.
Consequently, a new invariant solution of Eq. (1), written for constants $c_1 = -1/2$, $c_2 = 0$, takes the particular form:

$$u(t,x,y) = -\frac{x}{t}\left(y - \frac{x}{2t}\right) - \frac{1}{2}\left(y - \frac{x}{t}\right)^2 + \rho(t). \quad (51)$$

If we take into consideration the operator $X_3 + X_g = \frac{\partial}{\partial y} + g(t)\frac{\partial}{\partial u}$ and applying a similar way, one arrives at the following solution:

$$u(t,y) = g(t) y + r(t), \quad (52)$$

with $g(t)$ and $r(t)$ arbitrary functions.
As an example, for the choices $g(t) = \cos t$, $r(t) = \sin t$, the solution has the form plotted in Figure 2.

### 4.2. Particular solution from conservation laws

We will construct particular solutions of Eq. (1) by adding to this equation the differential constraints:

$$C^1 = C^1(x,y),\ C^2 = C^2(t,y),\ C^3 = C^3(t,x), \quad (53)$$

ISSN:0976-0172

Journal of Bioscience And Technology
www.jbstonline.com

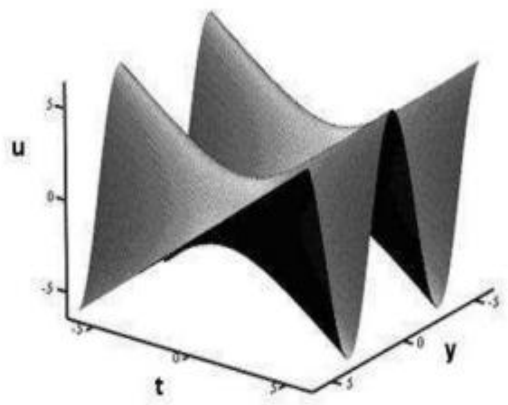


Figure 2. A plot of the periodic Rossby wave described by (52) with $g(t) = \cos t$ and $r(t) = \sin t$.

where the components of the conserved vector $C^i$, $i = 1, 2, 3$ are given by the expressions (32).
Thereby, a particular solution of the analyzed model provided by the conserved vector (32) are described by the system:

$$\begin{aligned} \Delta u_t + J(u, \Delta u) + \beta u_x &= 0, \\ D_t C^1 &= 0, \\ D_x C^2 &= 0, \\ D_y C^3 &= 0. \end{aligned} \tag{54}$$

This system is solved using the Maple program. The result is a periodic-type solution of the form:

**Concluding remarks**

In this paper we used two important approaches for finding exact solutions of the 2D inviscid barotropic nondivergent vorticity equation (1), namely the Lie symmetry and Ibragimov's approaches. Using the Lie symmetry algebra (13), three types of invariant similarity solutions generated by 1D subalgebra were pointed out. More precisely, we considered the subalgebras generated by $X_1$, $X_3 + X_f$, $X_3 + X_g$ and we found the associated solutions (46), (51), (52). These solutions were derived by solving the reduced PDEs (44), (49) written with respect to the appropriate invariant similarity variables. The solutions given show us that, in real atmospheric observations (in a background zonal basic wind), the stream function u may have not only linear shears, but also nonlinear shears. Another result of this paper is represented by the proof of strictly self-adjointness of the analyzed model, a feature which is essential in applying Ibragimov's method. Further, the construction of conservation laws for all symmetry operators from the basis (13) were investigated. In that direction, the thorough calculations proved that only the dilation group admitted by (1) generates a non-trivial conservation law, described by the conserved vector (32). The translation group and the infinite symmetry transformations involving two arbitrary functions of time, provide trivial conservation laws. The solution (55) corresponding to concrete expressions of the non-trivial conserved vector mentioned above, was also obtained. As this solution possesses a stable localized structure, it is a Rossby solitary wave.

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